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Multidimensional Single-Tone Frequency Estimation Based on QR Decomposition

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ABSTRACT In this paper, the problem of multidimensional single-tone frequency estimation of sinusoids embedded in white Gaussian noise is investigated. By extracting the two-dimensional (2-D) slice matrices from the multidimensional data, we construct a covariance matrix associated with only one dimension, from which the corresponding frequency is estimated with the utilization of a QR decomposition based iterative method. The frequencies of the remaining dimensions are then obtained following similar procedures. Moreover, the mean square error of the estimated frequencies is devised. The computer simulations are also included to evaluate the performance of the proposed method by comparing with the several state-of-the-art algorithms and Cramér–Rao lower bound.

INDEX TERMS Frequency estimation, QR factorization, parameter estimation.

I. INTRODUCTION

The problem of frequency estimation for multidimensional sinusoids has been an important research topic due to its important applications in engineering and areas such as radar, sonar, speech analysis, astronomy, array signal processing and nuclear magnetic resonance [1]–[8]. Various high-resolution frequency estimation methods have been proposed to solve this problem. The main concerns of handling multidimensional signals are reducing the computation complexity and reducing the dimension of signals. Such as the multidimensional folding (MDF) [9], the improved MDF (IMDF) [10], Unitary ESPRIT [11], MUSIC [12], [13], and RARE [8] methods, these methods solve the frequency estimation problem by first reducing the dimension of multidimensional signals. While HOSVD [14] algorithm directly utilizes tensor decomposition technique to solve the multidimensional harmonic retrieval problem, but the subsequent joint diagonalization algorithm takes much time in computation in order to avoid the problem of parameters pairing. In recent years, a computationally efficient method utilizing subspace and projection separation approaches (SPSA) [15] is proposed. Reference [16] proposes an algorithm whose perspective idea is to rearrange the multidimensional sampling arrays into a series of 2-D matrices, which is then utilized to construct a matrix from which the 2-D parameters can be estimated from its eigenvalues and eigenvectors. A tensor principal-singular-vector utilization for modal analysis (TPUMA) algorithm is developed in [17] for multidimensional harmonic retrieval. An N-D ESPRIT algorithm is proposed based on low rank decomposition of multilevel Hankel matrices formed by the multidimensional signal in [18], and its computational complexity is reduced by truncated singular value decomposition (SVD). In [19], an efficient sparse estimation approach (N-D Sparse) is developed for multidimensional harmonic retrieval. Most of the proposed methods can handle both multi-tone and single-tone frequency estimation problems. While [20] proposes a correlation-based method for single-tone frequency estimation of multidimensional data.

Inspired by the idea of reducing the data dimension by rearranging the multidimensional sampling arrays into a series of 2-D matrices [15], [16], in this paper, we contribute an accurate single-tone frequency estimation for multidimensional signal. By extracting two-dimensional (2-D) slice matrices from the multidimensional data, we construct a covariance matrix associated with only one dimension. Then the QR decomposition is employed to the covariance matrix, and an iterative procedure operates on the first row of upper triangular matrix $\mathbf{R}$ of the QR factorization is utilized to

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estimate the corresponding frequency. The frequencies of the remaining dimensions are estimated following a similar procedure as well.

The rest of this paper is organized as follows. In Section II, we state the problem and derive the proposed algorithm. In Section III, the performance analysis of proposed algorithm in term of theoretical mean square error (MSE) is formulated. And in Section IV, numerical simulations are included to evaluate the performance of the proposed algorithm by comparing with the correlation-based methods C-1 and C-2 [20], IMDF method [10], TPUMA method [17], N-D ESPRIT algorithm [18], and N-D Sparse algorithm [19] as well as the Cramér-Rao lower bound (CRLB) [21]. Finally, conclusions are drawn in Section V.

Notation: scalars, vectors and matrices are denoted by italic, bold lower-case and bold upper-case symbols, respectively. The angle of a is represented as $\angle(a)$. The derivative of a function $f(a)$ with respect to a is $f'(a)$. The estimate of $\mathbf{A}$ is $\hat{\mathbf{A}}$ and its rank is rank($\mathbf{A}$), while $^{\mathrm{T}}$, $^{\mathrm{H}}$, ^{A}st, and $^{-1}$ are the transpose, conjugate transpose, complex conjugate, inverse, respectively. The mth element of A is $[A]_m$ and the (m, n) entry of $\mathbf{A}$ is either denoted by $[\mathbf{A}]_{m,n}$ or $a_{m,n}$. The $\mathbf{I}_M$ symbolizes the $M \times M$ identity matrix and $0_{M\times N}$ represents the $M \times N$ zero matrix. Toeplitz(a, b^{T}) is a Toeplitz matrix with first column a and first row b^{T}. $\mathbb{E}$ is the expectation operator. Finally, $\mathbb{C}$ is used to represent the sets of complex numbers.

II. ALGORITHM DEVELOPMENT

A. SIGNAL MODEL

The observed single-tone multidimensional signal model is

$$x_{m_1,m_2,\cdots,m_R} = s_{m_1,m_2,\cdots,m_R} + n_{m_1,m_2,\cdots,m_R}, \quad m_r = 1, 2, \quad \cdots, \ M_r, \quad r = 1, 2, \ \cdots, \ R, \tag{1}$$

where

$$s_{m_1,m_2,\cdots,m_R} = \gamma \prod_{r=1}^{R} e^{j\omega_r m_r}, \tag{2}$$

is the noise-free signal with $R \geq 3$ being the number of sinusoids. γ is the unknown complex-valued amplitude, $\omega_r \in (-\pi, \pi)$ is the unknown frequency in the rth dimension. $n_{m_1,m_2,\cdots,m_R}$ is the additive zero-mean complex white Gaussian noise with unknown variance σ_n^2. Our task is to find the frequencies ω_r from the given $M = \prod_{r=1}^{R} M_r$ samples of $x_{m_1,m_2,\cdots,m_R}$.

B. ALGORITHM DEVELOPMENT

We first rewrite the signal model in tensor form [14], [15]

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}}, \tag{3}$$

where $\{\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{N}}\} \in \mathbb{C}^{M_1\times M_2\times\cdots\times M_R}$, $[\boldsymbol{\mathcal{X}}]_{m_1,m_2,\cdots,m_R} = x_{m_1,m_2,\cdots,m_R}$, $[\boldsymbol{\mathcal{S}}]_{m_1,m_2,\cdots,m_R} = s_{m_1,m_2,\cdots,m_R}$, and $[\boldsymbol{\mathcal{N}}]_{m_1,m_2,\cdots,m_R} = n_{m_1,m_2,\cdots,m_R}$.

Direct manipulations on tensor data demand a high computational complexity [15]. Following the idea in [15] to reduce the dimension of $\boldsymbol{\mathcal{X}}$, we consider the multidimensional sampling array as the collection of 2-D slice matrices by defining $\boldsymbol{\mathcal{X}}_{r_1,r_2}$ as follows

$$\boldsymbol{\mathcal{X}}_{r_1,r_2} = \{\boldsymbol{\mathcal{X}}(m_1, \cdots, m_{r_1-1}, :, m_{r_1+1}, \cdots, m_{r_2-1}, :, m_{r_2+1}, \cdots, m_R)\}, \tag{4}$$

where

$$m_r = 1, 2, \cdots, M_r, \quad r = 1, 2, \cdots, R \quad \text{and} \quad r \neq 1, 2.$$
$$\left[\boldsymbol{\mathcal{X}}(m_1, \cdots, m_{r_1-1}, :, m_{r_1+1}, \cdots, m_{r_2-1}, :, m_{r_2-1}, \cdots, m_R)\right]_{m_{r_1},m_{r_2}} = x_{m_1,m_2,\cdots,m_R}. \tag{5}$$

Following similar idea, $\boldsymbol{\mathcal{S}}_{1,r}$ is a noise-free 2-D matrix set of $\boldsymbol{\mathcal{S}}$, then $\boldsymbol{\mathcal{S}}(:, \cdots, m_{r-1}, :, m_{r+1}, \cdots, m_R)$ can be written as [15]

$$\boldsymbol{\mathcal{S}}(:, \cdots, m_{r-1}, :, m_{r+1}, \cdots, m_R) = \mathbf{g}_1 \lambda_{1,r} \mathbf{g}_r^{\mathrm{H}}, \tag{6}$$

which contains the frequency information corresponding to the first and rth dimension, where

$$\mathbf{g}_r = \left[a_r, a_r^2, \cdots, a_r^{M_r}\right], \tag{7}$$
$$a_1 = e^{j\omega_1}, \quad a_r = e^{-j\omega_r}, \ r = 2, 3, \cdots, R, \tag{8}$$
$$\lambda_{1,r} = \gamma \prod_{i=2\mathbf{I}\neq r}^{R} e^{j\omega_i m_i}. \tag{9}$$

To extract the signal subspace from the 2-D data matrix set, we compute the covariance matrix of matrix set $\boldsymbol{\mathcal{S}}_{1,2}$, denoted by $\hat{\mathbf{Z}}_1$,

$$\hat{\mathbf{Z}}_1 = \frac{M_1 M_2}{M} \sum_{m_3=1}^{M_3} \sum_{m_4=1}^{M_4} \cdots \sum_{m_R=1}^{M_R} \boldsymbol{\mathcal{S}}(:, :, m_3, \cdots, m_R) \cdot \boldsymbol{\mathcal{S}}^{\mathrm{H}}(:, :, m_3, \cdots, m_R). \tag{10}$$

Then the expected value of $\hat{\mathbf{Z}}_1$, denoted by $\mathbf{Z}_1$, has the form

$$\begin{aligned}\mathbf{Z}_1 &= \frac{M_1 M_2}{M} \sum_{m_3=1}^{M_3} \sum_{m_4=1}^{M_4} \cdots \sum_{m_R=1}^{M_R} \mathbf{g}_1 \lambda_{1,2} \mathbf{g}_2^{\mathrm{H}} \cdot \mathbf{g}_2 \lambda_{1,2}^{*} \mathbf{g}_1^{\mathrm{H}} \\ &= \frac{M_1 M_2}{M} \mathbf{g}_1 \left(\sum_{m_3=1}^{M_3} \sum_{m_4=1}^{M_4} \cdots \sum_{m_R=1}^{M_R} \lambda_{1,2} \mathbf{g}_2^{\mathrm{H}} \cdot \mathbf{g}_2 \lambda_{1,2}^{*} \right) \mathbf{g}_1^{\mathrm{H}}.\end{aligned} \tag{11}$$

By denoting

$$\beta_1 = \frac{M_1 M_2}{M} \sum_{m_3=1}^{M_3} \sum_{m_4=1}^{M_4} \cdots \sum_{m_R=1}^{M_R} \lambda_{1,2} \mathbf{g}_2^{\mathrm{H}} \cdot \mathbf{g}_2 \lambda_{1,2}^{*} \in \mathbb{C}, \tag{12}$$

we have

$$\mathbf{Z}_1 = \beta_1 \mathbf{g}_1 \mathbf{g}_1^{\mathrm{H}}, \tag{13}$$

where $\mathbf{Z}_1$ contains only one dimension's frequency information, then $\mathbf{Z}_1$ can be decomposed by eigenvalue decomposition to obtain the signal subspace then use subspace based method [15] to estimate the corresponding frequency.

On the other hand, it is easy to deduce that the elements along the columns or rows of $\mathbf{Z}_1$ satisfy the LP property [22], [23]. That is, the LP coefficients of row are characterized by $\mathbf{g}_1$. This implies that

$$z_{m_1,n_1} + c z_{m_1,n_1-1} = 0, \quad m_1 = 1, 2, \cdots, M_1, \quad n_1 = 2, 3, \cdots, M_1, \tag{14}$$

where z_{m_1,n_1} are the elements of $\mathbf{Z}_1$, $c = -e^{j\omega_1}$ is the LP coefficient [24], the rank of matrix $\mathbf{Z}_1$ is 1 for single-tone frequency.

Then $\mathbf{Z}_1$ can be factorized using QR decomposition [22], [23], [25] as

$$\mathbf{Z}_1 = \mathbf{QR}, \tag{15}$$

where $\mathbf{Q} \in \mathbb{C}^{M_1 \times M_1}$ is an orthonormal matrix and $R \in \mathbb{C}^{M_1 \times M_1}$ is an upper triangular matrix. Denoting $\mathbf{Q} = \begin{bmatrix}\mathbf{q}_1 \ \mathbf{q}_2 \ \cdots \ \mathbf{q}_{M_1}\end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} r_1 \ r_2 \ \cdots \ r_{M_1}\end{bmatrix}^{\mathrm{T}}$, we have $\mathbf{q}_i^{\mathrm{H}}\mathbf{q}_j = 0$ if $i \neq j$, and otherwise it is equal to 1, while $r_2, r_3, \cdots, r_{M_1}$ are all equal to $\mathbf{0}_{M_1 \times 1}$ since rank$(\mathbf{Z}_1) = 1$. We further let $\mathbf{Z}_1 = \begin{bmatrix}\mathbf{z}_1 \ \mathbf{z}_2 \ \cdots \ \mathbf{z}_{M_1}\end{bmatrix}$ and $\mathbf{z}_1 = \begin{bmatrix} z_{1,1} \ z_{2,1} \ \cdots \ z_{M_1,1}\end{bmatrix}^{\mathrm{T}}$, which results in $R_1 = \begin{bmatrix}\mathbf{q}_1^{\mathrm{H}}\mathbf{z}_1 \ \mathbf{q}_1^{\mathrm{H}}\mathbf{z}_2 \ \cdots \ \mathbf{q}_1^{\mathrm{H}}\mathbf{z}_{M_1}\end{bmatrix}^{\mathrm{T}}$. Analogously to (14), R_1 has the same LP property as in the row of $\mathbf{Z}_1$

$$[r_1]_l + c[r_1]_{l-1} = 0, l = 2, 3, \cdots, M_1. \tag{16}$$

We can express (16) in matrix form as

$$A r_1 = \mathbf{u}c - \mathbf{f} = \mathbf{0}_{(M_1-1)\times 1}, \tag{17}$$

where

$$A = \text{Toeplitz}\left(\begin{bmatrix} c \ \mathbf{0}_{1\times(M_1-2)}\end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} c \ 1 \ \mathbf{0}_{1\times(M_1-2)}\end{bmatrix}\right), \tag{18}$$

$$\mathbf{u} = \begin{bmatrix}[r_1]_1 \ [r_1]_2 \ \cdots \ [r_1]_{M_1-1}\end{bmatrix}^{\mathrm{T}}, \tag{19}$$

$$\mathbf{f} = -\begin{bmatrix}[r_1]_2 \ [r_1]_3 \ \cdots \ [r_1]_{M_1}\end{bmatrix}^{\mathrm{T}}. \tag{20}$$

When noise is present, we perform QR decomposition on the constructed covariance matrix $\mathbf{Z}_1$ using $\boldsymbol{\mathcal{X}}_{1,r}$ to obtain $\mathbf{Q}$ and $\mathbf{R}$. Then (13) is rewritten as

$$\mathbf{Z}_1 = \beta_1 \mathbf{g}_1 \mathbf{g}_1^{\mathrm{H}} + \sigma_n^2 \mathbf{I}_{M_1}, \tag{21}$$

where $\sigma_n^2 \mathbf{I}_{M_1}$ is the noise. (21) can be expressed into a matrix form

$$\mathbf{Z}_1 = \mathbf{S}_1 + \mathbf{N}_1, \tag{22}$$

where $\mathbf{S}_1 = \beta_1 \mathbf{g}_1 \mathbf{g}_1^{\mathrm{H}}, \mathbf{N}_1 = \sigma_n^2 \mathbf{I}_{M_1}$. Hence, (17) will no longer be a zero vector and we denote it by $\mathbf{e} = \mathbf{u}c - \mathbf{f}$, with $\mathbf{e} \in \mathbb{C}^{(M_1-1)\times 1}$. For the linear model of (17), the weighted least squares (WLS) estimate of c is [4]

$$\hat{c} = \arg\min_{c} \mathbf{e}^{\mathrm{H}}\mathbf{W}\mathbf{e} = \left(\mathbf{u}^{\mathrm{H}}\mathbf{W}\mathbf{u}\right)^{-1}\mathbf{u}^{\mathrm{H}}\mathbf{W}\mathbf{f}, \tag{23}$$

where $\mathbf{W}$ is a symmetric weighting matrix and its optimal choice is derived using the covariance for $\mathbf{e}$ [25]

$$\mathbf{W} = \sigma_n^2 \left[\mathbb{E}\left\{\mathbf{e}\mathbf{e}^{\mathrm{H}}\right\}\right]^{-1} = (\mathbf{A}\mathbf{A}^{\mathrm{H}})^{-1}, \tag{24}$$

and its optimal choice using the covariance for $\mathbf{e}$ is derived as follows:

TABLE 1. Estimation algorithm for ω_r.

(i) Generate the 2-D slice matrices using the multidimensional signal $\boldsymbol{\mathcal{X}}_{1,r}$
(ii) Construct the covariance matrix $\mathbf{Z}_r$
(iii) Perform the QR factorization based algorithm on the covariance matrix $\mathbf{Z}_r$
(iv) Estimate the frequency ω_r using the WLS based iterative method.

Using (22), the property of QR decomposition, and $\mathbf{A}\mathbf{S}^{\mathrm{T}} = \mathbf{0}_{(M_1-1)\times M_1}$, we can express $\mathbf{e}$ as

$$\mathbf{e} = \mathbf{A}r_1 = \mathbf{A}\mathbf{Z}_1^{\mathrm{T}}\mathbf{q}_1^* = \mathbf{A}(\mathbf{S}_1 + \mathbf{N}_1)^{\mathrm{T}}\mathbf{q}_1^* = \mathbf{A}\mathbf{N}_1^{\mathrm{T}}\mathbf{q}_1^*. \tag{25}$$

Hence we have:

$$\mathbf{e}\mathbf{e}^{\mathrm{H}} = \mathbf{A}\mathbf{N}_1^{\mathrm{T}}\mathbf{q}_1^*\mathbf{q}_1^{\mathrm{T}}(\mathbf{N}_1^{\mathrm{T}})^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}. \tag{26}$$

As matrix $\mathbf{Q}$ is an orthonormal matrix, $\mathbf{q}_1^*\mathbf{q}_h^{\mathrm{T}} = I_{M_1}$ when $h = 1$, otherwise it is equal to $\mathbf{0}_{M_1\times M_1}$, and $\mathbb{E}\left\{\mathbf{N}_1^{\mathrm{T}}(\mathbf{N}_1^{\mathrm{T}})^{\mathrm{H}}\right\} = \sigma_w^2 I_{M_1}$, the expected value of (26) is then

$$\mathbb{E}\{\mathbf{e}\mathbf{e}^{\mathrm{H}}\} = \sigma_n^2 \mathbf{A}\mathbf{A}^{\mathrm{H}}. \tag{27}$$

Substituting (27) into (24) yields the weighting matrix expression.

The frequency estimate in the first dimension is

$$\hat{\omega}_1 = \angle(\hat{c}). \tag{28}$$

As in the ideal case, the weight is a function of the unknown c, we solve $\hat{c}$ to relax (23) in an iterative manner. We initialize $\mathbf{D}$ using the identity matrix I_{M_1-1}, then iterate between (23) and (24), and terminate the algorithm after $\mathcal{I}$ iterations.

Since frequency pairing is not needed for single-tone frequency estimation, we then employ the first row of $\mathbf{R}$ in the QR factorization of $\mathbf{Z}_r, r = 2, 3, \cdots, R$, which are constructed following (4) to (11), to compute $\omega_r, r = 2, 3, \cdots, R$ in a similar manner as shown in Table 1.

III. PERFORMANCE ANALYSIS

In this section, we derive the mean and MSE expressions for the frequency estimate to analyze the performance of the proposed algorithm. As seen in Section II that the proposed QR decomposition based method estimates the frequency of each dimension in a separate manner. Then the data model for each dimension is in fact 1-D which we can apply [26] to carry out the performance analysis. The basic idea is to utilize (23), with which $\hat{c}$ satisfies

$$\hat{c} = (\mathbf{u}^{\mathrm{H}}\mathbf{W}\mathbf{u})^{-1}\mathbf{u}^{\mathrm{H}}\mathbf{W}\mathbf{f}, \tag{29}$$

and we construct a function $f(\check{c})$ based on (29) as follows

$$\begin{aligned} f(\check{c}) &= \mathbf{u}^{\mathrm{H}}\mathbf{W}\mathbf{u}\check{c} - \mathbf{u}^{\mathrm{H}}\mathbf{W}\mathbf{f} \\ &= \mathbf{u}^{\mathrm{H}}\mathbf{W}(\mathbf{u}\check{c} - \mathbf{f}), \end{aligned} \tag{30}$$

such that $f(\hat{c}) = 0$. With sufficiently large SNR and data size, $\hat{c}$ will have a value close to c. Using the Taylor's series to expand $f(\hat{c})$, we obtain

$$0 = f(\hat{c}) \approx f(c) + f^{'}(c)(\hat{c} - c), \tag{31}$$

where $f'(c)$ is the first derivative of $f(\check{c})$ evaluated at $\check{c} = c$, then we have

$$\hat{c} \approx c - \frac{f(c)}{f'(c)}, \tag{32}$$

and we obtain $\mathbb{E}\{\hat{c}\} \approx c$ [27]. On the other hand, the MSE of $\hat{c}$ is [27]

$$\begin{aligned} \mathrm{MSE}(\hat{c}) &= \mathbb{E}\{(\hat{c}-c)(\hat{c}-c)^*\} \\ &= \frac{\sigma_n^2}{\mathbf{u}^{\mathrm{H}}\mathbf{W}\mathbf{u}}. \end{aligned} \tag{33}$$

According to [28] the MSE of ω_1 with $\mathrm{SNR} = \sigma_s^2/\sigma_n^2$, where $\sigma_s^2 = \frac{\left(\sum_{m_1=1}^{M_1}\cdots\sum_{m_R=1}^{R}|s_{m_1,m_2,\cdots,m_R}|^2\right)}{M}$, are

$$\mathrm{MSE}(\hat{\omega}_1) \approx \frac{\mathrm{MSE}(\hat{c})}{2|c|^2} = \frac{\sigma_n^2}{2\mathbf{u}^{\mathrm{H}}\mathbf{W}\mathbf{u}} \approx \frac{6}{M(M_1^2-1)\mathrm{SNR}}. \tag{34}$$

Following the same idea above, we obtain the general form of MSE for ω_r is

$$\mathrm{MSE}(\hat{\omega}_r) \approx \frac{6}{M(M_r^2-1)\mathrm{SNR}}, \tag{35}$$

which is approximate to the CRLB [21] of frequency for the rth dimension. The derivatives of mean and MSE indicate that our proposed algorithm is effective and accurate.

IV. NUMERICAL EXAMPLES

Computer simulations have been conducted to evaluate the proposed QR decomposition based algorithm. For simplicity, we use three-dimensional (3-D) sinusoids in the presence of white Gaussian noise in the simulation. In each dimension, we terminate the iterative procedure after two iteration (i.e., $\mathcal{I} = \in$) because no significant improvement was achieved for more iterations. The MSE in comparison is computed using $\mathbb{E}\{(\omega_r - \hat{\omega}_r)^2\}$, with the CRLB to measure the performance of the proposed method. The correlation-based methods C-1 and C-2 in [20], and IMDF algorithm in [10] TPUMA method [17], N-D ESPRIT algorithm [18], and N-D Sparse algorithm [19] are included for comparison. Note that a Fast N-D ESPRIT algorithm is also developed in [18], but its performance is similar with the N-D ESPRIT algorithm, so we only compare the computation time of the Fast N-D ESPRIT algorithm in the computation comparison experiment. We scale the noise to produce different SNR conditions. The amplitude is set to $\gamma = 1$. All results provided are averages of 500 independent runs. Our simulations are performed using MATLAB R2017b on a system with 1.70 GHz intel Xeon CPU E5-2609 and 32 GB RAM, under a 64-bit Windows 10 operating system.

In the first experiment, we set the sinusoidal frequency $\omega_1 = 0.3\pi$, $\omega_2 = 0.05\pi$, $\omega_3 = 0.9\pi$, which are the frequencies of the first dimension, second dimension, and third dimension, respectively. The dimensions of the 3-D data are $M_1 = M_2 = M_3 = 15$. The frequency MSE results of the three dimensions are plotted in Figures 1, 2, and 3, respectively. As seen from all these three figures, the proposed

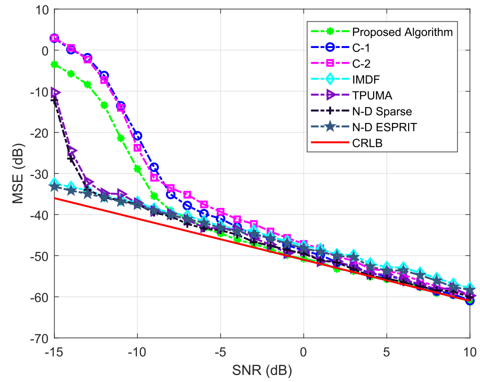

FIGURE 1. MSE versus SNR with $M_1 = M_2 = M_3 = 15$ (1^{st} dimension).

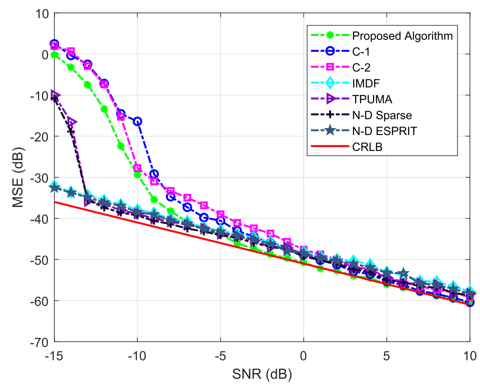

FIGURE 2. MSE versus SNR with $M_1 = M_2 = M_3 = 15$ (2^{nd} dimension).

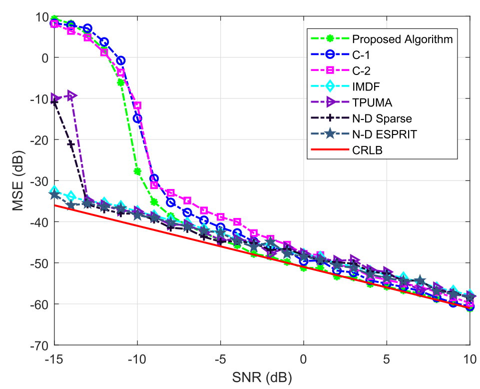

FIGURE 3. MSE versus SNR with $M_1 = M_2 = M_3 = 15$ (3^{rd} dimension).

algorithm can achieve optimal performance at sufficiently high SNRs, and better than the correlation-based methods C-1 and C-2, which aligns with our analysis in Section 3. On the other hand, even though the MSE of the other methods in comparison is lower than the proposed method at small SNRs, the proposed method outperforms the other methods when the SNR is larger than −5 dB.

In the next experiment, we compare the performance of the proposed method with different numbers of samples. We set $M_1 = 15$, $M_2 = 14$, $M_3 = 13$. The other data settings remain the same as that in experiment 1. The result is plotted in Figures 4-6. It is seen that the MSEs of the proposed

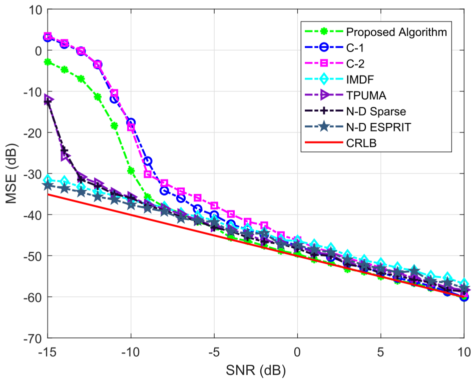

FIGURE 4. MSE versus SNR with $M_1 = 15$, $M_2 = 14$, $M_3 = 13$ (1^{st} dimension).

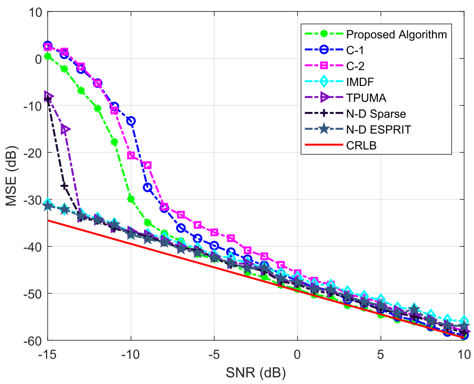

FIGURE 5. MSE versus SNR with $M_1 = 15$, $M_2 = 14$, $M_3 = 13$ (2^{nd} dimension).

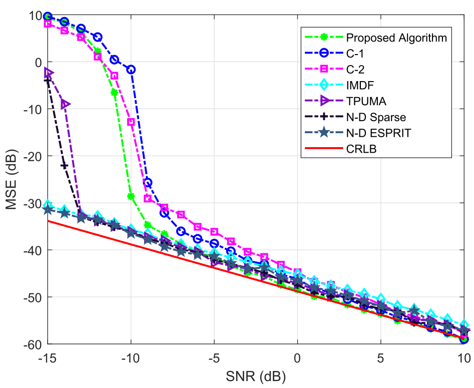

FIGURE 6. MSE versus SNR with $M_1 = 15$, $M_2 = 14$, $M_3 = 13$ (3^{rd} dimension).

algorithm always attain the CRLB at certain SNRs for different data length, and the performance of the proposed algorithm are similar with that in experiment 1.

In the third experiment, we investigate the performance of the proposed method when the numbers of samples are varying from 10 to 20, i.e., M_1 (M_2 and M_3) varies from 10 to 20. The SNR is fixed at SNR = 5 dB. For simplicity, we compare the average MSE (AMSE) of the three dimension's frequencies. The other data settings are the same as that in experiment 1. From Figure 7, it is seen that the AMSE of the

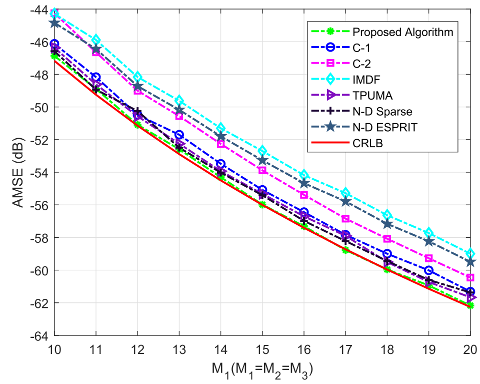

FIGURE 7. AMSE versus $M_1 (M_1 = M_2 = M_3)$.

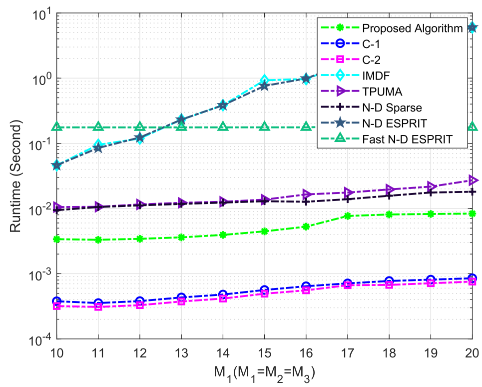

FIGURE 8. Computation time versus $M_1 (M_1 = M_2 = M_3)$.

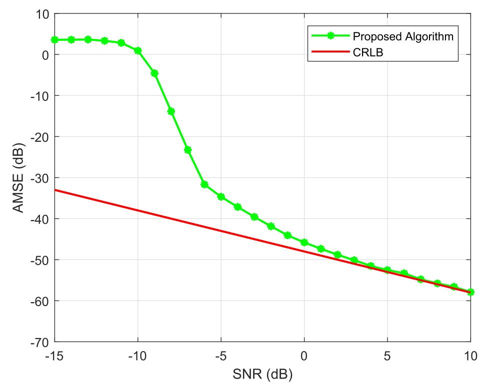

FIGURE 9. AMSE versus SNR with $M_1 = M_2 = M_3 = 15$ for 2-tone case.

proposed algorithm always attain the CRLB and outperforms all the other methods.

In the forth experiment, we plot the average computational time of the proposed algorithm and the other methods versus different M with $M_1 = M_2 = M_3$ in Figure 8. M_1 varies from 10 to 20. We fix the SNR at 5 dB, the other data settings remain the same as in the first test. We also included the average computational time of Fast N-D ESPRIT method in [18]. It is seen that even though the proposed approachs cost more time than the correlation-based methods C-1 and C-2, it is more computationally efficient than the IMDF, N-D Sparse, and N-D ESPRIT methods, although the latter have better threshold behaviors as shown in previous simulation results. These simulation results indicate that the proposed method can achieve accurate estimate with less complexity compared with IMDF, N-D Sparse, N-D ESPRIT, and Fast N-D ESPRIT methods.

Although the purpose of this paper is to solve the problem of multidimensional single-tone frequency estimation, it is also possible to extend the proposed algorithm to multi-tone case with the utilization of an existing frequency pairing method. But due to that the matrix R of the QR decomposition is an upper triangular matrix, the accuracy of the estimate will be reduced with directly use of the proposed algorithm. To investigate the performance of directly use of the proposed algorithm, we compare the MSE of frequencies estimated from a 3-D signal with 2-tone frequencies with CRLB. The frequencies of the 3-D signal is $[\omega_{11}\ \omega_{12}] = [0.3\pi\ 0.5\pi]$, $[\omega_{21}\ \omega_{22}] = [0.05\pi\ 0.23\pi]$, $[\omega_{31}\ \omega_{32}] = [0.4\pi\ 0.9\pi]$, and $[\gamma_1\ \gamma_2] = [1\ 2]$. We compare the AMSE of the three dimension's frequencies for simplicity. The result is plotted in Figure 9. It is seen that the performance of the proposed method is reduced compared with that of the single-tone case, the AMSE attains the CRLB when SNR is larger than 2 dB.

V. CONCLUSION

To estimate the frequency of multidimensional signals embedded in additive white Gaussian noise, an accurate approach based on QR decomposition is addressed. As we only consider single-tone multidimensional signals, no pairing procedure is needed. Both theoretical development and simulation results show that the proposed method can provide optimal estimation performance when the SNR is sufficiently high compared with existing state-of-the-art algorithms.

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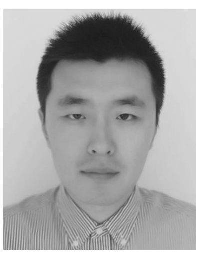

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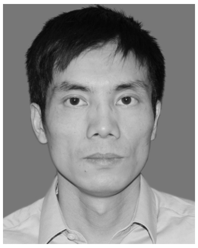

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