Fast Rank-Revealing QR Factorization for Two-Dimensional Frequency Estimation

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***Abstract*— It is well known that the singular value decomposition (SVD), as the best rank-revealing factorization, furnishes the best rank-k approximation to a dense matrix with expensive computational cost of $\mathcal{O}(M^2N + N^2M + \min(M,N)^3)$. Moreover, it is hard to implement in parallel, which challenges the memory storage in wireless communications data-driven system. In this letter, a fast rank-revealing technique, namely, bilateral random projections (BRP) with $\mathcal{O}(MN)$ operations, is exploited for two-dimensional (2-D) frequency estimation of a complex sinusoid in noisy environment. Based on the resulting data matrix, whereafter, two-stage QR factorization frequency estimation method with the weighted least squares (WLS) as solver, is proposed to reduce the computational complexity of those SVD-based frequency estimators. Simulation results demonstrate the efficiency of the proposed algorithm in comparison with several frequency estimation approaches and the Cramér-Rao lower bound (CRLB) as benchmark.**

***Index Terms*— Two-dimensional frequency estimation, bilateral random projections, rank-revealing QR factorization, weighted least squares.**

I. Introduction

FREQUENCY estimation is a key technique for the receiver design in wireless communications, such as carrier frequency offset, and is of broad interest, with wide applications arising in radar signal processing, speech analysis, smart grid stability, health assessment of living trees, to name just a few [1]–[3]. Recently, the Doppler and frequency shifts are struggled to avoid the deterioration in the quantum satellite communication performance of orthogonal frequency-division multiplexing (OFDM)-based systems, the interested reader is referred to [4], [5]. Therefore, frequency estimation is fertile research ground that merits further investigation on the fast and accurate approaches for reliable wireless communications data-driven system.

A. Prior Works

The iterative quadratic maximum likelihood (IQML) method [6] achieves an optimum estimation at the expense of heavy computational complexity due to a multidimensional search to find the global maximum of the maximum likelihood (ML) cost function, which is not suitable for real-time applications. To reduce the IQML's implementation cost, [7] is proposed based on a relaxation algorithm for the ML formulation to produce a near-optimum solution. Conventional subspace-based algorithms, for instance, multiple signal classification (MUSIC) [8] and estimation of signal parameters via rotational invariance techniques (ESPRIT) [9], are also utilized for frequency estimation, which can achieve a very high estimation accuracy with low complexity. Motivated by the fact that the left and right principal singular vectors contain the frequencies in the first and second dimensions, respectively, the principal-singular-vector utilization for modal analysis (PUMA) [10], is proposed with singular value decomposition (SVD) and weighted least squares (WLS) techniques. On the other hand, numerous efforts [11], [12] have been made to compute the frequency of a sinusoidal by interpolating on the discrete Fourier transform (DFT) coefficients. However, most of them are iterative methods or rely on the peak DFT coefficients or its neighbors to estimate the frequency. These approaches suffer from high computational load or lower SNR threshold problems. Recently, 2-D root-MUSIC is proposed for joint angle and Doppler estimation in colocated multiple-input multiple-output (MIMO) radar system [13].

B. Contributions

We propose a two-stage QR factorization 2-D frequency estimation method based on the fast rank-revealing bilateral random projections (BRP) technique. Motivated by that one-dimensional (1-D) frequency information is embedded in the upper triangular matrix $\mathbf{R} \in \mathbb{C}^{M\times N}$ of QR factorization on the resulting 2-D data matrix $\mathbf{Y} \in \mathbb{C}^{M\times N}$, a novel signal subspace is obtained from the first row entries of $\mathbf{R}$, which is the null space of We apply the WLS approach to estimate the frequency from the obtained subspace. The other dimension's frequency is estimated following a similar procedure. The contributions of our work are briefly summarized as follows:

1) To speed up the implementation, BRP with $\mathcal{O}(MN)$ operations is utilized to approximate the dense data matrix. Moreover, because of the computationally demanding SVD with $\mathcal{O}(M^2N + N^2M + \min(M,N)^3)$ operations, two-stage QR factorization for 2-D frequency estimation is applied to further reduce the computational complexity, where QR factorization only costs $\mathcal{O}(M^2N - N^3/3)$ complexity.

2) The rank-revealing property of BRP is first approved. It is verified by simulation result that the proposed method with BRP performs better than that without BRP.

3) The proposed method achieves comparable performance to the Cramér-Rao lower bound (CRLB) and outperforms several 2-D frequency estimation approaches, where the mean square error (MSE) of the estimate is also derived.

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II. Algorithm Development

In this work, we consider the problem of 2-D frequency estimation of a complex sinusoid with additive Gaussian noise, modeled as following discrete formulation [14]:

$$x_{m,n} = s_{m,n} + w_{m,n}, \quad m = 1, 2, \cdots, M, \quad n = 1, 2, \cdots, N, \; M \geq N, \tag{1}$$

where the ground-truth signal is $s_{m,n} = \gamma e^{j(\mu m + \nu n)}$ with $\gamma \in \mathbb{C}$ as the complex amplitude of signal. Without loss of generality, we assume that an additive noise $w_{m,n}$ is distributed as Gaussian process with mean zero and unknown variance σ^2. The unknown parameters $\{\mu, \nu\} \in (-\pi, \pi)$ are the 2-D frequencies to be estimated from the observed measurements $\{x_{m,n}\}$ with MN samples. For ease of presentation, (1) is rewritten in matrix form as:

$$\mathbf{X} = \mathbf{S} + \mathbf{W}. \tag{2}$$

The elements of $\mathbf{S}$ are denoted as $[\mathbf{S}]_{m,n} = s_{m,n} = \gamma e^{j(\mu m + \nu n)}$, where $[\cdot]_{m,n}$ denotes the element of a matrix. To clearly show the relationships of the elements in the rows and columns of $\mathbf{S}$, Lemma 1 is introduced as follows[1]:

Lemma 1: Unfolding the noise-free signal matrix $\mathbf{S}$, it has the form of

$$\mathbf{S} = \begin{bmatrix} \gamma e^{j\mu} e^{j\nu} & \gamma e^{j\mu} e^{j2\nu} & \cdots & \gamma e^{j\mu} e^{jN\nu} \\ \gamma e^{j2\mu} e^{j\nu} & \gamma e^{j2\mu} e^{j2\nu} & \cdots & \gamma e^{j2\mu} e^{jN\nu} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma e^{jM\mu} e^{j\nu} & \gamma e^{jM\mu} e^{j2\nu} & \cdots & \gamma e^{jM\mu} e^{jN\nu} \end{bmatrix}.$$

Since each columns or rows of $\mathbf{S}$ are characterized by $\{e^{j\mu}\}$ and $\{e^{j\nu}\}$, respectively, it is easy to observe that the entries along its corresponding columns or rows satisfy the linear prediction (LP) property [14].

For a dense $M \times N$ matrix $\mathbf{X}$ in (2), SVD, as the best rank-k approximation to $\mathbf{X}$, can be used to determine the rank of matrix but at the cost of highly computational complexity. In this work, a fast low-rank approximation method, BRP [15] is exploited for the approximation to $\mathbf{X}$. Given k BRP of $\mathbf{X}$, i.e., $\mathbf{L}_1 := \mathbf{X}\boldsymbol{\Gamma}_1$ and $\mathbf{L}_2 := \mathbf{X}^T\boldsymbol{\Gamma}_2$, $\mathbf{X}$ is approximated by

$$\mathbf{Y} = \mathbf{L}_1(\boldsymbol{\Gamma}_2^T \mathbf{L}_1)^{-1}\mathbf{L}_2^T, \tag{3}$$

where $\mathbf{Y} \in \mathbb{C}^{M\times N}$ is the data matrix obtained from $\mathbf{X}$ after BRP, $\boldsymbol{\Gamma}_1 \in \mathbb{C}^{N\times k}$ and $\boldsymbol{\Gamma}_2 \in \mathbb{C}^{M\times k}$. For the dense matrix $\mathbf{X}$, $\mathcal{O}(MNk)$ operations are required to obtain BRP and $\mathcal{O}(k^2(2N+k)+MNk)$ operations are sufficed to get $\mathbf{Y}$, where the computational complexity of BRP is much less than that of SVD-based approximation methods. Next, we show the rank revealing property of BRP.

Lemma 2: We observe that $\mathbf{Y}$ is a reduced form of $\mathbf{X}$ from (3). Therefore, the interlacing inequalities of singular values between $\mathbf{Y}$ and $\mathbf{X}$, lead to

$$\sigma_i(\mathbf{Y}) \leq \sigma_i(\mathbf{X}). \tag{4}$$

Denote that $\mathbf{M} := (\boldsymbol{\Gamma}_2^T\mathbf{L}_1)^{-1}$. We factorize $\mathbf{Y}$ with partitioned $\mathbf{M}$, that is:

$$\mathbf{Y} = \mathbf{L}_1 \begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \mathbf{L}_2^T. \tag{5}$$

[1]In the absence of noise, $\mathbf{Y} \approx \mathbf{X} = \mathbf{S}$. Both $\mathbf{Y}$ and $\mathbf{X}$ satisfy the LP property. Motivated by that, 1-D information w.r.t. μ or ν can be decoupled.

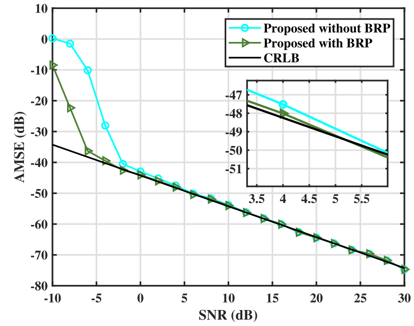

Fig. 1. AMSE of μ and ν in comparison with/without BRP.

Since $\mathbf{M}$ is obtained by the BRP of $\mathbf{Y}$, we have:

$$\sigma_i(\mathbf{M}_{11}) \leq \sigma_i(\mathbf{Y}). \tag{6}$$

From (4) and (6), we obtain $\sigma_i(\mathbf{M}_{11}) \leq \sigma_i(\mathbf{X})$. Moreover, the ℓ_2-norm of $\mathbf{M}_{11}$ is small. Hence, these facts suggest that BRP is a rank-revealer, which helps to improve the proposed method wherein it is also verified by the simulation result in Fig. 1. As seen from the figure, the proposed algorithm with BRP enjoys better threshold than that without BRP. Now, to achieve the automatic pairing of frequencies, QR factorization is first applied to the novel data matrix $\mathbf{Y}$, which is expressed as the product of an unitary matrix $\mathbf{Q} \in \mathbb{C}^{M\times M}$ and an rank-revealing upper triangular matrix $\mathbf{R} \in \mathbb{C}^{M\times N}$:

$$\mathbf{Y} = \mathbf{QR} = \mathbf{Q}\begin{bmatrix} r_{11} & \mathbf{r}_{12} \\ \mathbf{0} & \mathbf{R}_{22} \end{bmatrix}, \tag{7}$$

where $\mathbf{R}$ contains the information of frequency ν, r_{11} is the first element of $\mathbf{R}$, $\mathbf{r}_{12}$ is the first row of $\mathbf{R}$ except the first element, and $\mathbf{R}_{22} \in \mathbb{C}^{(M-1)\times(N-1)}$ is the rest of $\mathbf{R}$ except r_{11} and $\mathbf{r}_{12}$. As the Frobenius norm of $\mathbf{r}_{12}$ is small, we use the first row of $\mathbf{R}$ to obtain the basis of the noise space. The first row of $\mathbf{R}$, namely, $\bar{\mathbf{r}} := [r_{11} \; \mathbf{r}_{12}]$, is the null space of $\mathbf{Y}$. Then,

$$\bar{\mathbf{r}}\underline{\mathbf{g}} = \begin{bmatrix} r_{11} & \mathbf{r}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = 0. \tag{8}$$

where $\underline{\mathbf{g}} := [\mathbf{g}_1^T \; \mathbf{g}_2^T]^T$ is the orthogonal space of $\mathbf{Y}$ and $\mathbf{g}_1 = -r_{11}^{-1}\mathbf{r}_{12}\mathbf{g}_2$ as $r_{11} \neq 0$. Therefore,

$$\underline{\mathbf{g}} = \begin{bmatrix} -r_{11}^{-1}\mathbf{r}_{12} \\ \mathbf{I}_{N-1} \end{bmatrix} \mathbf{g}_2 := \mathbf{P}\mathbf{g}_2. \tag{9}$$

To ensure the columns of $\mathbf{P}$ being orthonormal, the orthogonal projection onto $\mathbf{P}$ is applied. We get $\mathcal{P}\perp := \mathbf{P}(\mathbf{P}^H\mathbf{P})^{-1}\mathbf{P}^H$. Hence, the novel signal subspace is computed from $\mathbf{V}_s = \mathbf{I}_M - \mathcal{P}\perp$. By assumed that $\bar{\mathbf{v}}_s := [v_1 \; v_2 \; \cdots \; v_N]$ is the first row of $\mathbf{V}_s$, then

$$\mathbf{v}_{s1}c - \mathbf{v}_{s2} = \mathbf{0}_{(N-1)\times 1}, \tag{10}$$

in which $\mathbf{v}_{s1} := [v_1 \; v_2 \; \cdots \; v_{N-1}]$, $\mathbf{v}_{s2} := [v_2 \; v_3 \; \cdots \; v_N]$, and c is the LP factor. In matrix form, (10) is rewritten as:

$$\mathbf{A}\tilde{\mathbf{v}}_s = \mathbf{0}_{(N-1)\times 1}, \tag{11}$$

where $\mathbf{A} = \text{Toeplitz}\left(\left[-c\ \mathbf{0}_{1\times(N-2)}\right]^T, \left[-c\ 1\ \mathbf{0}_{1\times(N-2)}\right]\right)$. In presence of noise, $\mathbf{A}\tilde{\mathbf{v}}_s = \mathbf{e}$, where $\mathbf{e}$ is the error vector. Following [14], the estimate of c is computed by

$$\min_c \mathbf{e}^H \Phi \mathbf{e} = \min_c (\mathbf{v}_{s1} c - \mathbf{v}_{s2})^H \mathbf{\Phi} (\mathbf{v}_{s1} c - \mathbf{v}_{s2}), \quad (12)$$

where the symmetric weight matrix $\mathbf{\Phi}$ is derived from the covariance of $\mathbf{e}$ [10]. Taking advantage of QR property, $\mathbf{A}\mathbf{S}^T = \mathbf{0}_{(N-1)\times M}$, we express $\mathbf{e}$ as:

$$\mathbf{e} = \mathbf{A}\tilde{\mathbf{v}}_s = \mathbf{A}\mathbf{Y}^T \underline{\mathbf{q}}^* \approx \mathbf{A}(\mathbf{S}+\mathbf{W})^T \underline{\mathbf{q}}^* = \mathbf{A}\mathbf{W}^T \underline{\mathbf{q}}^*, \quad (13)$$

where * denotes the conjugate operator and $\underline{\mathbf{q}}$ is the first column of matrix $\mathbf{Q}$. Here we use $\mathbf{Y}^T$ in (13) because the frequency information of second dimension is contained in $\mathbf{R}$ of the QR factorization. Hence we have:

$$\mathbf{e}\mathbf{e}^H = \mathbf{A}\mathbf{W}^T \underline{\mathbf{q}}^* \underline{\mathbf{q}}^T (\mathbf{W}^T)^H \mathbf{A}^H. \quad (14)$$

As $\mathbf{Q}$ is an orthonormal matrix, $\underline{\mathbf{q}}^* \underline{\mathbf{q}}^T = \mathbf{I}_M$, and $\mathbb{E}\left\{\mathbf{W}^T(\mathbf{W}^T)^H\right\} = \sigma^2 \mathbf{I}_N$, then $\mathbb{E}\{\mathbf{e}\mathbf{e}^H\} = \sigma^2 \mathbf{A}\mathbf{A}^H$. Then $\mathbf{\Phi} = \sigma^2 \left[\mathbb{E}\{\mathbf{e}\mathbf{e}^H\}\right]^{-1} = (\mathbf{A}\mathbf{A}^H)^{-1}$. On the other hand, we assume $\mathcal{J}(c) = (\mathbf{v}_{s1} c - \mathbf{v}_{s2})^H \mathbf{\Phi} (\mathbf{v}_{s1} c - \mathbf{v}_{s2})$, then $\hat{c}$ is computed by differentiating $\mathcal{J}(c)$ with respect to (w.r.t.) c and setting the resultant expression to zero:

$$\mathcal{J}'(c) = 2c\mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1} - 2\mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s2} = 0. \quad (15)$$

We obtain:

$$\hat{c} = \frac{\mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s2}}{\mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1}}. \quad (16)$$

We calculate $\hat{c}$ using (12) and update $\mathbf{\Phi}$ with $c = \hat{c}$ in an iterative manner, then the final estimate of frequency in the second dimension is:

$$\hat{\nu} = \angle(\hat{c}), \quad (17)$$

On the other hand, μ is estimate in a similar manner based on the QR factorization of $\mathbf{Y}^T$. We summarized the whole procedure as follows:

i) Apply BRP on the dense data matrix $\mathbf{X}$, then perform the QR factorization on $\mathbf{Y}$ to obtain $\mathbf{R}$.

ii) Use the first row of $\mathbf{R}$ to construct $\mathcal{P}\perp$.

iii) Obtain the signal subspace $\mathbf{V}_s$ from $\mathcal{P}\perp$, then conduct the WLS procedure from (7)-(17) to obtain the estimate of ν.

iv) Perform QR factorization on $\mathbf{X}^T$ and repeat ii)-iii) to get the estimate of μ.

III. Performance Analysis

To further verify the accuracy of the proposed method, the MSE of the estimate are derived.

From (16), we construct:

$$\mathcal{F}(\hat{c}) = \mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1} \hat{c} - \mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s2}. \quad (18)$$

Using the Taylor series expansion,

$$\begin{aligned}\mathcal{F}(\hat{c}) &\approx \mathcal{F}(c) + \mathcal{F}'(c)(\hat{c} - c) \\ &= \mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1} c - \mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s2} + \mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1} (\hat{c} - c). \end{aligned} \quad (19)$$

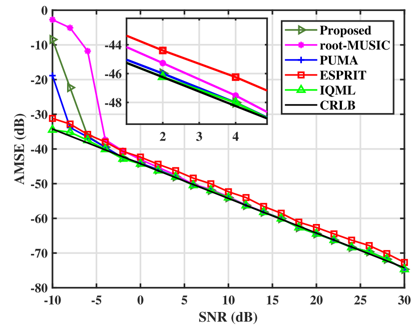

Fig. 2. AMSE of μ and ν versus SNR.

where $\mathcal{F}'(c)$ is the first derivative of $\mathcal{F}(\hat{c})$ evaluated at $\hat{c} = c$. Let (19) be zero and we have:

$$\hat{c} \approx c - \frac{\mathcal{F}(c)}{\mathcal{F}'(c)} = c - \frac{\mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1} c - \mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s2}}{\mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1}}, \quad (20)$$

and we obtain $\mathbb{E}\{\hat{c}\} \approx c$.[2] The MSE of $\hat{c}$ is:

$$\text{MSE}(\hat{c}) = \mathbb{E}\{(\hat{c} - c)(\hat{c} - c)^*\} = \frac{\sigma_n^2}{\mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1}}. \quad (21)$$

Based on [16], the MSE of $\hat{\nu}$ with $\text{SNR} = |\gamma|^2/\sigma^2$, is:

$$\begin{aligned}\text{MSE}(\hat{\nu}) &\approx \frac{\text{MSE}(\hat{c})}{2|c|^2} = \frac{\sigma_n^2}{2\mathbf{v}_{s1}^H \mathbf{\Phi} \mathbf{v}_{s1}} \\ &\approx \frac{6\sigma^2}{MN(N^2-1)|\gamma|^2} \approx \frac{6}{MN(N^2-1)\text{SNR}}. \end{aligned} \quad (22)$$

Similarly, the MSE of $\hat{\mu}$ is:

$$\text{MSE}(\hat{\mu}) \approx \frac{6}{MN(M^2-1)\text{SNR}}. \quad (23)$$

IV. Numerical Examples

In this section, computer simulations are conducted to evaluate the performance of the proposed method in white Gaussian noise, compared with root-MUSIC [13], PUMA [10], ESPRIT [9], and IQML [6] algorithms as well as CRLB [6]. The MSEs of μ and ν are defined as $\mathbb{E}\{(\hat{\mu} - \mu)^2\}$ and $\mathbb{E}\{(\hat{\nu} - \nu)^2\}$, respectively. We compare the averaged MSE (AMSE), which is the average of μ and ν's MSEs in the simulations. We scale the noise matrix to produce different SNR conditions where $\text{SNR} = |\gamma|^2/\sigma^2$ with $\gamma = 1$. Unless stated otherwise, $M = N = 20$. Our simulations are performed on the MATLAB R2017b of 64-bit Windows 10 operating system with 1.70 GHz intel Xeon CPU E5-2609 and 32 GB RAM.

In the first test, the sinusoidal parameters are assigned as $\mu = 0.15\pi$ and $\nu = 0.35\pi$. As shown in Fig. 2, the proposed algorithm achieves optimal performance when SNR is larger than -5 dB. All compared methods can approach the CRLB at smaller SNR except the root-MUSIC and ESPRIT, while the ESPRIT and IQML have better threshold performance than others.

[2]With sufficiently large SNR and data size, $\hat{c}$ will have a value close to c. It means that the proposed method is unbiased estimator, which is also demonstrated by our simulation results.

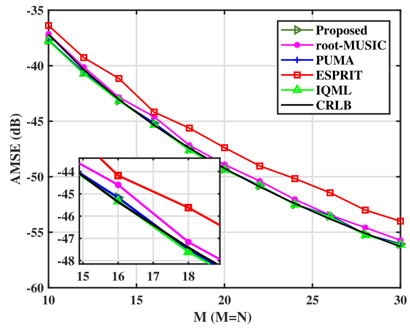

Fig. 3. AMSE of μ and ν versus M.

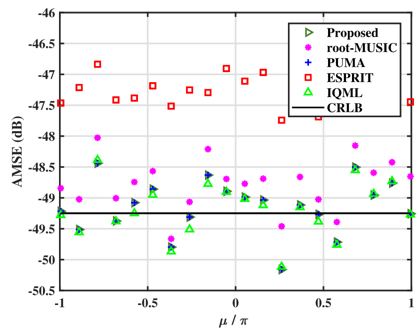

Fig. 4. AMSE of μ and ν versus μ.

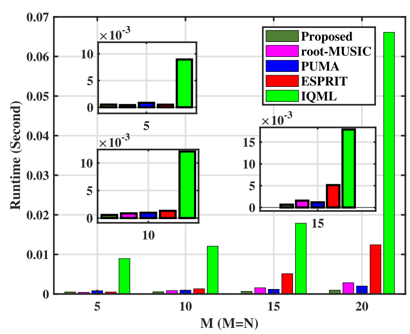

Fig. 5. Computation time versus M.

In the second test, we study the AMSE of the proposed method when the signal size $M(N)$ ranges from 10 to 30, as shown in Fig. 3. We observe that the proposed method enjoys comparable performance with the compared algorithms except the ESPRIT.

In the third test, we investigate the AMSE of μ and ν when they varies from $-\pi$ to π, where SNR $= 5$ dB. The results are plotted in Fig. 4. The proposed method is comparable to the PUMA and IQML but is superior to the root-MUSIC and ESPRIT.

The last test compares the computational complexity, wherein the computer runtime is shown in Fig. 5 and the complexity analysis is tabulated in Table I. We can see that the proposed algorithm enjoys the lowest computation complexity among all methods in comparison. That is to say, the proposed algorithm is a computationally simple solution with minimum achievable MSE at sufficiently high SNR conditions, although its threshold behavior is inferior to its counterparts.

TABLE I
COMPUTATIONAL COMPLEXITY FOR DIFFERENT ALGORITHMS

Algorithms	Computational complexity
Proposed	$\mathcal{O}(N^2M - NM^2 + M^3 + MN)$
root-MUSIC	$\mathcal{O}(N^2 + M^2 + M^3 + N^3)$
PUMA	$\mathcal{O}(N^2M + NM^2 + M^3)$
ESPRIT	$\mathcal{O}(M^2N^2)$
IQML	$\mathcal{O}(M^3N^3)$

V. CONCLUSION

A computationally efficient frequency estimator for 2-D complex sinusoid is proposed, where two-stage processes are designed based on QR factorization to construct the novel signal subspace w.r.t. 1-D frequency. Benefit from the rank-revealing property of BRP, it contributes to reduce the computational complexity and improve the accuracy of the estimator. It is worth noting that the proposed method can be easily extended to the application of multi-tone frequency estimation of 2-D complex sinusoid.

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